Lorentz invariance and confined noncommutativity

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Abstract

Lorentz invariance in a noncommutative space-time structure is retrieved by mass normalization of a fermion in free motion. We provide a noncommutative extension of momentum conservation law which includes spin polarization change. A detail structure of noncommutative parameter $\theta^{\mu\nu}$ is given. We also describe a noncommutative effect on classical compton scattering and a process to calculate weak mixing angle with an initial result, $\sin^2\theta_W = 1/5$.

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I. INTRODUCTION

Lorentz symmetry breaking of noncommutative field theory [1–3] has been considered as an intrinsic property of the noncommutative space-time structure by virtue of noncommutative parameter $\theta^{\mu\nu}$ [4] and extensively studied in the literature [5–8]. A general consequence naturally follows that momentum does not conserve in the space-time manifold [9–11].

On the other hand, the Seiberg-Witten map [3] allows one to take a specific case, such as the non-commutativity is restricted inside of a fermion if there is a correspondence between the dynamics of D0-branes and confined quarks [12]. In the case, the invariant mass of the noncommutative fermion under noncommutative boost transformation must be exactly same as the one from commutative Lorentz transformation of commutative fermion. Therefore, we may raise a question what kinds of representation is suitable to describe the Lorentz invariant confined noncommutative structure in free motion. In other words, the argument implies that the geometric definition

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1}$$

alone may have a limit to represent physical nature of the noncommutative fermion in Minkowski space-time manifold.

In this paper, we investigate the confined noncommutative structure by imposing spin to momentum. As a result, we find a Lorentz invariant, confined noncommutative space-time mass structure with a detail form of $\theta^{\mu\nu}$. In section 2, a self-consistent framework is given with a noncommutative extension of momentum conservation. By noting that the confined noncommutative structure should hold for whole range of momentum $\beta = [0, 1]$, in section 3, a modified classical compton scattering formula is presented to test the approach at low energy scale. In section 4, as an application to weak energy scale, we provide a simple calculation of weak mixing angle with a primitive result, $\sin^2\theta_W = 1/5$. Finally, some concluding remarks are given.

II. CONFINED NONCOMMUTATIVE STRUCTURE

To include spin on momentum in the absence of non-Abelian gauge coupling, we may go back to the isospin definition of W. Heisenberg [13]. Since the nucleon in free motion can not distinguish the momentum and spin polarization in the absence of electromagnetic interaction, the state at different noncommutative space-time point may be written as

$$\begin{pmatrix} P'^{\mu} \\ s'^{\mu} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} P^{\mu} \\ s^{\mu} \end{pmatrix} \tag{2}$$

The noncommutatively transformed momentum and spin need to obey Lorentz covariance so that the four unknown constants, which are naturally related to isospin by virtue of two states of nucleon, can be reduced to a single free parameter as in special relativity. Since the new definition contains isospin matrices, it automatically provides the proposed confined noncommutative space-time structure.

From the usual definition of momentum and spin, $P^{\mu} = mc\gamma(1, \boldsymbol{\beta})$ and $s^{\mu} = (s^0, \mathbf{s})$, the normalization and the orthogonal conditions follow

$$P_{\mu}P^{\mu} = m^2c^2 \ , \qquad s_{\mu}s^{\mu} = -1 \ , \qquad P_{\mu}s^{\mu} = 0 \tag{3}$$

Since the noncommutative transformation Eq. (2) need to obey Lorentz transformation, it is convenient to keep same notations as in special relativity

$$\begin{aligned} P'^{\mu} &= \gamma(\epsilon)(P^{\mu} + \epsilon P^{\mu}_{s}) \\ P'^{\mu}_{s} &= \gamma(\epsilon)(\epsilon P^{\mu} + P^{\mu}_{s}) \end{aligned} \tag{4}$$

where the notation $P^{\mu}_{s} \equiv mcs^{\mu}$ with $\lambda mc = h$ is introduced to keep the same unit of momentum. h is the Planck constant. $\gamma(\epsilon) = \cosh\vartheta$ and $\gamma(\epsilon)\epsilon = \sinh\vartheta$ with $\gamma(\epsilon) = 1/\sqrt{1-\epsilon^2}$ follow the same Lorentz transformation in special relativity with the rotational angle ϑ for finite transformation. The variable ϵ can be easily identified as a constant velocity for the noncommutative transformation and it must have a connection to isospin. We may draw a map

$$\epsilon = \tanh\vartheta = [-1, 1] \quad \longleftrightarrow \quad \beta = \tanh\omega = [0, 1] \tag{5}$$

as non-Abelian and Abelian velocities in noncommutative transformation. With the definitions, the noncommutative transformation generates Lorentz invariant quantities

$$\begin{aligned} P'_{\mu}P'^{\mu} &= P_{\mu}P^{\mu} = m^2c^2 \\ P'_{s\mu}P'^{\mu}_{s} &= P_{s\mu}P^{\mu}_{s} = -m^2c^2 \end{aligned} \tag{6}$$

The second term reflects a typical spin structure, the double transformation. There is only one free parameter ϵ in this structure. It is simple enough to reproduce the commutative state in the limit of the non-Abelian velocity

$$\begin{aligned} P'^{\mu} &\xrightarrow{\epsilon\to 0} P^{\mu} \\ P'^{\mu}_{s} &\xrightarrow{\epsilon\to 0} P^{\mu}_{s} \end{aligned} \tag{7}$$

From the map, we may extend the non-Abelian velocity to 3-dimensional space

$$\begin{aligned} \beta &\longrightarrow \boldsymbol{\beta} \\ \epsilon &\longrightarrow \boldsymbol{\epsilon} \end{aligned} \tag{8}$$

with the 3-dimensional noncommutative transformation

$$
\begin{aligned}
P'^{\mu} &= \gamma(\epsilon)(P^{\mu} + \boldsymbol{\epsilon} \cdot \mathbf{P}_s^{\mu}) \\
P_s'^{\mu} &= \gamma(\epsilon)(\boldsymbol{\epsilon} \cdot \mathbf{P}^{\mu} + P_s^{\mu})
\end{aligned} \tag{9}
$$

Both the spin polarization and the momentum get the additional two degrees of freedom by following the unknown direction of 3-dimensional non-Abelian velocity $\boldsymbol{\epsilon} = \hat{\imath}\epsilon_1 + \hat{\jmath}\epsilon_2 + \hat{k}\epsilon_3$.

$$
\begin{aligned}
\mathbf{P}^{\mu} &= \hat{\imath}P_1^{\mu} + \hat{\jmath}P_2^{\mu} + \hat{k}P_3^{\mu} \\
\mathbf{P}_s^{\mu} &= \hat{\imath}P_{s1}^{\mu} + \hat{\jmath}P_{s2}^{\mu} + \hat{k}P_{s3}^{\mu}
\end{aligned} \tag{10}
$$

The 3D transformation still reproduce the commutative state in the limit $\boldsymbol{\epsilon} \to 0$, but does not reproduce the Lorentz invariance. This is a general consequence of typical noncommutative space-time structure [14,1]. However, in this confined approach, we know from the map what is missing. Since we could call the $\mathbf{P}^{\mu}$ and $\mathbf{P}_s^{\mu}$ as the non-Abelian vector momentum and the non-Abelian spin polarization vector, separately, the vector part of Lorentz transformation is the missing one. For pure boost transformation, we find

$$
\begin{aligned}
\mathbf{P}_s'^{\mu} &= \gamma\left(\mathbf{P}_s^{\mu} + \frac{\gamma - 1}{\epsilon^2}\boldsymbol{\epsilon}\boldsymbol{\epsilon} \cdot \mathbf{P}_s^{\mu} + \gamma\boldsymbol{\epsilon}P^{\mu}\right) \\
\mathbf{P}'^{\mu} &= \gamma\left(i\mathbf{P}^{\mu} + \frac{\gamma\epsilon - i}{\epsilon^2}\boldsymbol{\epsilon}\boldsymbol{\epsilon} \cdot \mathbf{P}^{\mu} + \gamma\hat{\epsilon}P_s^{\mu}\right)
\end{aligned} \tag{11}
$$

Transformation of isovector spin polarization will be derived in this section. Together with the isoscalar part, we define the second rank noncommutative 4-momentum and 4-spin polarization

$$
\begin{aligned}
P^{\alpha\mu} &= (P^{\mu}, \mathbf{P}_s^{\mu}) \\
P_s^{\alpha\mu} &= (P_s^{\mu}, \mathbf{P}^{\mu})
\end{aligned} \tag{12}
$$

Thus, boost transformation reads

$$
\begin{aligned}
{P'}_{\alpha\mu}P'^{\alpha\mu} &= P_{\alpha\mu}P^{\alpha\mu} = P_{\mu}P^{\mu} - \mathbf{P}_{s\mu} \cdot \mathbf{P}_s^{\mu} \neq m^2c^2 \\
{P_s}'_{s\alpha\mu}P_s'^{\alpha\mu} &= P_{s\alpha\mu}P_s^{\alpha\mu} = P_{s\mu}P_s^{\mu} - \mathbf{P}_{\mu} \cdot \mathbf{P}^{\mu} \neq -m^2c^2
\end{aligned} \tag{13}
$$

It shows the non-Abelian velocity $\boldsymbol{\epsilon}$ transformation is Lorentz invariant. Problem is still remained since it is not invariant for $\boldsymbol{\epsilon}$=0. We will continue it after fixing the unknown structure of $\boldsymbol{\epsilon}$ first.

Until now we presume $\boldsymbol{\epsilon}$ has a non-Abelian structure, the matrix structure, in free motion. To find the structure, it is necessary to turn on gauge field through minimal coupling in the 3-dimensional noncommutative transformation Eq. (9). For simplicity, we consider only the momentum transformation. There are two ways to turn on the minimal coupling with the noncommutative approach. One is to start with the commutative minimal coupling state and then to transform both the momentum and gauge field by the non-Abelian velocity $\boldsymbol{\epsilon}$, the other is to start with the non-Abelian free motion state and then to generate the minimal coupling to the noncommutatively transformed state. Both ways produce same result

$$
\begin{aligned}
P'^{\mu} &= \gamma\left[(P^{\mu} - \tfrac{1}{2}qA^{\mu}) + \boldsymbol{\epsilon} \cdot (\mathbf{P}_s^{\mu} - \tfrac{1}{2}q\mathbf{A}_s^{\mu})\right] \\
P'^{\mu} &= \gamma\left(P^{\mu} + \boldsymbol{\epsilon} \cdot \mathbf{P}_s^{\mu}\right) - \tfrac{1}{2}q\gamma\left(A^{\mu} + \boldsymbol{\epsilon} \cdot \mathbf{A}_s^{\mu}\right)
\end{aligned} \tag{14}
$$

where the charge q is supposed to independent on the noncommutative transformation and $\gamma = \gamma(\epsilon)$. By comparing the SU(2)xU(1) Yang-Mills gauge field, we can easily draw the three important consequences of noncommutative structure

$$
\begin{aligned}
\boldsymbol{\epsilon} &= \epsilon \boldsymbol{\tau} \\
g' &= \gamma(\epsilon) q \\
g &= \gamma(\epsilon) \epsilon q \\
A^{\alpha\mu} &= (A^{\mu}(x, x_s), \mathbf{A}_s^{\mu}(x, x_s))
\end{aligned} \tag{15}
$$

where $\boldsymbol{\tau}$ is the isospin matrix. The first result is that the supposed non-Abelian velocity $\boldsymbol{\epsilon}$ becomes isospin velocity which gives a consistent base to construct a noncommutative space-time structure. Second result is that the only one free parameter ϵ exists in the theory by connecting the gauge field definition and the space-time definition. The third one is that the gauge fields acquire the additional coordinate dependence, x_s^{μ}, which is related on the spin polarization and the direction of isospin velocity $\boldsymbol{\epsilon}$. Reducing to commutative state is simply done by the single parameter setting, $\epsilon \to 0$, the zero of isospin velocity.

Now, two important physical variables are appeared

$$
\begin{aligned}
P^{\alpha\mu} &= (P^{\mu}, \mathbf{P}_s^{\mu}) \\
A^{\alpha\mu} &= (A^{\mu}(x, x_s), \mathbf{A}_s^{\mu}(x, x_s))
\end{aligned} \tag{16}
$$

Looking at the non-Abelian structure is always guaranteed by Lorentz transformation with the isospin velocity $\boldsymbol{\epsilon}$. So far the process does not provide on the detail structure of non-commutative space-time as in Eq. (1). We may find a clue from the two ingredients, (1) x_s^{μ} is related on spin polarization, (2) x_s^{μ} dependence of gauge field suggests $\mathbf{P}_s^{\mu}$ has some sort of **differential operator** in parallel with the relation of commutative position x^{μ} and commutative momentum $P^{\mu} = i\partial_x^{\mu}$. From the clue, we review the relations of spin polarization and momentum Eq. (3). Since the spin polarization can be expressed by

$$
P_s^{\mu} = mc \frac{1}{\sqrt{1 - (\hat{s} \cdot \boldsymbol{\beta})^2}} (\hat{s} \cdot \boldsymbol{\beta}, \hat{s}) \tag{17}
$$

the longitudinally polarized spin, $\hat{s} = \hat{\beta}$, becomes

$$
P_s^{\mu} = mc \frac{1}{\sqrt{1 - \beta^2}} (\beta, \hat{\beta}) = mc(\gamma\beta, \gamma\hat{\beta}) \tag{18}
$$

where $\gamma = \gamma(\beta) = 1/\sqrt{1 - \beta^2}$. When we note on the relations

$$
\begin{aligned}
\frac{1}{\gamma^2} \frac{d}{d\beta} \ \gamma &= \ \gamma\beta \\
\frac{1}{\gamma^2} \frac{d}{d\beta} \ \gamma\beta &= \ \gamma
\end{aligned} \tag{19}
$$

the longitudinally polarized spin can be transformed from momentum by the *differential operation*

$$
P_s^{\mu} = \frac{1}{\gamma^2} \frac{d}{d\beta} \ P^{\mu} \tag{20}
$$

Conversely, the momentum transits from the longitudinally polarized spin

$$
P^{\mu} = \frac{1}{\gamma^2} \frac{d}{d\beta} \ P_s^{\mu} \tag{21}
$$

The differential relations provide a concrete base to the conjecture Eq. (2) since the momentum and spin polarization are not independent on each other for velocity change. The orthogonal condition shows another meaning by the differential operation,

$$P_\mu P_s^\mu = P_\mu \frac{1}{\gamma^2}\frac{d}{d\beta}P^\mu = \frac{1}{\gamma^2}\frac{d}{d\beta}(\tfrac{1}{2}m^2c^2) = 0 \tag{22}$$

the independence on β-change of the Lorentz invariant quantity. We may interpret that the longitudinally polarized fermion in free motion has to be invariant both on space-time change ∂_x^μ and on velocity change $\frac{d}{d\beta}$ also. In other words, the right-handed and the left-handed fermions must be free of an axial force during the free motion state to keep Lorentz invariance. We may take a reverse interpretation such that if a fermion carries a portion of transverse polarization in addition to longitudinal polarization, then there is an axial force to keep the Lorentz invariance. It means when spin polarization is changed by an interaction process, the axial force is the source of change. The argument could apply to the case of momentum change equally by the relation Eq. (21). In general, spin related axial force may involve in collision process. It is interesting that an analysis to noncommutative e-p scattering process draws same conclusion [10]. It seems we have a powerful tool to define the noncommutative space-time structure dynamically. We continue to apply the tool on noncommutatively transformed states Eq. (4). Since there are two γ notations, it might be convenient to introduce another notations to distinguish the Abelian motion $\gamma(\beta)$ and the non-Abelian motion $\gamma(\epsilon)$ easily. Let's define

$$\begin{aligned} \epsilon_p &= \gamma(\epsilon) = 1/\sqrt{1-\epsilon^2} = \cosh\vartheta \\ \epsilon_p\epsilon &= \gamma(\epsilon)\epsilon = \epsilon/\sqrt{1-\epsilon^2} = \sinh\vartheta \\ \epsilon &= \tanh\vartheta = [-1,1] \end{aligned} \tag{23}$$

Then, the Eq. (4) becomes

$$\begin{aligned} P'^\mu &= \epsilon_p(P^\mu + \epsilon P_s^\mu) = \epsilon_p(1+\epsilon\frac{1}{\gamma^2}\frac{d}{d\beta})P^\mu \\ P_s'^\mu &= \epsilon_p(\epsilon P^\mu + P_s^\mu) = \epsilon_p(1+\epsilon\frac{1}{\gamma^2}\frac{d}{d\beta})P_s^\mu \end{aligned} \tag{24}$$

For 3-dimensional case, we may extend it as Eq. (9)

$$\begin{aligned} P'^\mu &= \epsilon_p(P^\mu + \boldsymbol{\epsilon}\cdot\mathbf{P}_s^\mu) = \epsilon_p(1+\boldsymbol{\epsilon}\cdot\frac{1}{\gamma^2}\nabla_\beta)P^\mu \\ P_s'^\mu &= \epsilon_p(\boldsymbol{\epsilon}\cdot\mathbf{P}^\mu + P_s^\mu) = \epsilon_p(1+\boldsymbol{\epsilon}\cdot\frac{1}{\gamma^2}\nabla_\beta)P_s^\mu \end{aligned} \tag{25}$$

with

$$\nabla_s \equiv \frac{1}{\gamma^2}\nabla_\beta = \frac{1}{\gamma^2}\left(\hat{\beta}\frac{\partial}{\partial\beta} + \hat{\theta}\frac{1}{\beta}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{\beta\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{26}$$

where we define $\hat{s}_\beta = \hat{\beta},\ \hat{s}_\theta = \hat{\theta},\ \hat{s}_\varphi = \hat{\varphi}$ from the spin polarization Eq. (17). Therefore, the non-Abelian spin polarization and the non-Abelian momentum can be written as a differential operator

$$
\begin{aligned}
\mathbf{P}_s^\mu &= \nabla_s P^\mu = \frac{1}{\gamma^2}\nabla_\beta P^\mu = \frac{1}{\gamma^2}\nabla_\beta \;\; i\partial_x^\mu \\
\mathbf{P}^\mu &= \nabla_s P_s^\mu = \frac{1}{\gamma^2}\nabla_\beta P_s^\mu
\end{aligned} \tag{27}
$$

The two variables dependence, x and β, is a crucial observation with regarding to the two variable dependence of gauge field $A^\mu(x, x_s)$. They should have a connection since they are originated from the same source, the spin polarization. For a moment, we need to get involving on the urgent questions on the Lorentz invariance, the orthogonality, and the physical meaning. The 3D differential operator ∇_s provides detail structures

$$
\begin{aligned}
\frac{1}{mc}\mathbf{P}_s^\mu &= \frac{1}{mc}\left(\hat{\beta}P_{s_\beta}^\mu + \hat{\theta}P_{s_\theta}^\mu + \hat{\varphi}P_{s_\varphi}^\mu\right) = \hat{\beta}\gamma(\beta, \hat{\beta}) + \hat{\theta}\frac{1}{\gamma}(0, \hat{\theta}) + \hat{\varphi}\frac{1}{\gamma}(0, \hat{\varphi}) \\
\frac{1}{mc}\mathbf{P}^\mu &= \frac{1}{mc}\left(\hat{\beta}P_\beta^\mu + \hat{\theta}P_\theta^\mu + \hat{\varphi}P_\varphi^\mu\right) = \hat{\beta}\gamma(1, \boldsymbol{\beta}) + \hat{\theta}\frac{1}{\gamma\beta}(0, \hat{\theta}) + \hat{\varphi}\frac{1}{\gamma\beta}(0, \hat{\varphi})
\end{aligned} \tag{28}
$$

We note on finding the de Broglie relation in the θ and φ components in the non-Abelian momentum.

$$
\begin{aligned}
\frac{\lambda}{mc}P_\theta^\mu &= \frac{h}{p}(0, \hat{\theta}) = \lambda s_\theta^\mu \\
\frac{\lambda}{mc}P_\varphi^\mu &= \frac{h}{p}(0, \hat{\varphi}) = \lambda s_\varphi^\mu
\end{aligned} \tag{29}
$$

with $p = mc\gamma\beta$. We may call the two 4-vectors as de Broglie 4-vectors. The noncommutative extension of orthogonal conditions are given by

$$
\begin{aligned}
&P \cdot \mathbf{P}_s = 0\,, \qquad P_{s\beta} \cdot P_{s\theta} = P_{s\theta} \cdot P_{s\varphi} = P_{s\varphi} \cdot P_{s\beta} = 0 \\
&P_s \cdot \mathbf{P} = 0\,, \qquad P_\beta \cdot P_\theta = P_\theta \cdot P_\varphi = P_\varphi \cdot P_\beta = 0
\end{aligned} \tag{30}
$$

It says that Lorentz invariance must hold for the spin polarization change both in gradient direction β, θ, φ and in successive changes of direction $\beta\theta, \theta\varphi, \varphi\beta$. The noncommutative extension of normalization conditions with $m = c = 1$ follow

$$
\begin{aligned}
&P \cdot P = 1\,, \quad P_{s\beta} \cdot P_{s\beta} = -1\,, \;\; P_{s\theta} \cdot P_{s\theta} = -1/\gamma^2\,, \;\; P_{s\varphi} \cdot P_{s\varphi} = -1/\gamma^2 \\
&P_s \cdot P_s = -1\,, \quad P_\beta \cdot P_\beta = 1\,, \;\; P_\theta \cdot P_\theta = -1/\gamma^2\beta^2\,, \;\; P_\varphi \cdot P_\varphi = -1/\gamma^2\beta^2
\end{aligned} \tag{31}
$$

The results implies that the noncommutative extension of momentum and spin polarization satisfies the extended orthogonal conditions. However, something is still missing since the extended normalization conditions depend on β.

$$
\begin{aligned}
P'_{\alpha\mu}P'^{\alpha\mu} &= m^2c^2\left[1 - \left(-1 - \frac{1}{\gamma^2} - \frac{1}{\gamma^2}\right)\right] \\
P'_{s\alpha\mu}P_s'^{\alpha\mu} &= m^2c^2\left[-1 - \left(1 - \frac{1}{\gamma^2\beta^2} - \frac{1}{\gamma^2\beta^2}\right)\right]
\end{aligned} \tag{32}
$$

we note that the direct application of the process to noncommutative space-time structure does not generate the invariance but it does in the limit $\beta \to 0$ for the noncommutative momentum $P'^{\alpha\mu}$. The invariance is necessary to make a theory independent on noncommutative space-time transformation.

We may find a clue from that the differential operator ∇_s gives detail structures to noncommutative momentum and spin. Since the differential operator comes from spin and acts only on velocity β, one may consider the boosted space-time to relate the operator to Pauli spin operator.

$$x^\mu(\beta) = \gamma(x^\mu + \beta x^\mu_\lambda) \tag{33}$$

with $x^\mu = (x^0, x)$ and $x^\mu_\lambda = (x, x^0)$. Then, from the observation that the two different operators, σ_1 and $\frac{1}{\gamma^2}\frac{d}{d\beta}$, to the transformed 4-vector $x^\mu(\beta)$ exhibit the same physical result,

$$\begin{aligned} \sigma_1 x^\mu(\beta) &= \gamma(x^\mu_\lambda + \beta x^\mu) \\ \frac{1}{\gamma^2}\frac{d}{d\beta}x^\mu(\beta) &= \gamma(\beta x^\mu + x^\mu_\lambda) \end{aligned} \tag{34}$$

we may connect the two operators with a working rule: (1) taking the relation between the Pauli spin matrix and the differential operator as an eigen equation, (2) finding out eigenfunctions for each different spin operations which gives same physical meaning on the two operatons,

$$\frac{1}{\gamma^2}\frac{d}{d\beta}\phi(\beta) = \sigma_1\phi(\beta) \tag{35}$$

The two states of eigenfunction $\phi(\beta)$ can be easily determined by integration to the specified spin state $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

$$\phi(\beta) = \begin{pmatrix} \phi_+(\beta) \\ \\ \phi_-(\beta) \end{pmatrix} = \begin{pmatrix} \gamma \\ \\ \gamma\beta \end{pmatrix} \tag{36}$$

The process can be extended to 3-dimensional $\boldsymbol{\beta}$-space as far as σ_1 extends to 3-dimensional $\boldsymbol{\sigma}$-space

$$\frac{1}{\gamma^2}\nabla_\beta\phi(\boldsymbol{\beta}) = \frac{1}{\gamma^2}\left(\hat{\beta}\frac{\partial}{\partial\beta} + \hat{\theta}\frac{1}{\beta}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{\beta\sin\theta}\frac{\partial}{\partial\varphi}\right)\phi(\boldsymbol{\beta}) = \boldsymbol{\sigma}\phi(\boldsymbol{\beta}) \tag{37}$$

with the eigen functions: the β-components,

$$\begin{aligned} \phi_{\beta+}(\boldsymbol{\beta}) &= Y^n_l(\theta', \varphi'), \qquad \cos\theta' = \beta \\ \phi_{\beta-}(\boldsymbol{\beta}) &= \frac{1}{1-i}\left(\frac{\sqrt{3}}{\gamma^2}\phi'_{\beta+} - \phi_{\beta+}\right) \end{aligned} \tag{38}$$

with a spherical harmonic function Y^n_l and a discrete velocity $\beta^2 = 1 - \frac{n^2-1}{l(l+1)}$, the θ-components,

$$\begin{aligned} \phi_{\theta+}(\boldsymbol{\beta}) &= e^{\theta\gamma^2\beta} \\ \phi_{\theta-}(\boldsymbol{\beta}) &= (1+i)(1+\sqrt{3/2})\phi_{\theta+} \end{aligned} \tag{39}$$

and the φ-components,

$$\begin{aligned} \phi_{\varphi+}(\boldsymbol{\beta}) &= e^{\varphi\gamma^2\beta\sin\theta} \\ \phi_{\varphi-}(\boldsymbol{\beta}) &= -\frac{\sqrt{2}}{1-i}\phi_{\varphi+} \end{aligned} \tag{40}$$

where $\sigma_\beta = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)$, $\sigma_\theta = \frac{1}{\sqrt{6}}(\sigma_1 + \sigma_2 - 2\sigma_3)$, $\sigma_\varphi = \frac{1}{\sqrt{2}}(\sigma_2 - \sigma_1)$, which satisfy all of anticommutative relations of spin matrices, are used. Since the eigen equations always determine the precise form of the state function of spin operators, the state function could be related to the noncommutaive structure of spin motion of fermion $\psi(x, x_s)$ and boson $A^\mu(x, x_s)$. Equivalently, the noncommutative momentum and spin polarization in second rank tensor Eq. (12) can be expressed in a compact form

$$\begin{aligned} P^{\alpha\mu} &= (P^\mu, \mathbf{P}_s^\mu) = (\sigma^0, \boldsymbol{\sigma})P^\mu = \sigma^\alpha P^\mu \\ P_s^{\alpha\mu} &= (P_s^\mu, \mathbf{P}^\mu) = (\sigma^0, \boldsymbol{\sigma})P_s^\mu = \sigma^\alpha P_s^\mu \end{aligned} \tag{41}$$

The transformation with isospin velocity $\boldsymbol{\epsilon}$ is given by

$$\begin{aligned} P^{\alpha\mu}(\boldsymbol{\epsilon}) &= (P^\mu(\boldsymbol{\epsilon}), \mathbf{P}_s^\mu(\boldsymbol{\epsilon})) = (\sigma^0(\boldsymbol{\epsilon}), \boldsymbol{\sigma}(\boldsymbol{\epsilon}))P^\mu = \sigma^\alpha(\boldsymbol{\epsilon})P^\mu \\ P_s^{\alpha\mu}(\boldsymbol{\epsilon}) &= (P_s^\mu(\boldsymbol{\epsilon}), \mathbf{P}^\mu(\boldsymbol{\epsilon})) = (\sigma^0(\boldsymbol{\epsilon}), \boldsymbol{\sigma}(\boldsymbol{\epsilon}^{-1}))P_s^\mu = \sigma^\alpha(\boldsymbol{\epsilon})P_s^\mu \end{aligned} \tag{42}$$

with

$$\begin{aligned} \sigma^0(\boldsymbol{\epsilon}) &= \epsilon_p(\sigma^0 + \boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}) \\ \boldsymbol{\sigma}(\boldsymbol{\epsilon}) &= \epsilon_p\left(\boldsymbol{\sigma} + \frac{\epsilon_p - 1}{\epsilon^2}\boldsymbol{\epsilon}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} + \epsilon_p\boldsymbol{\epsilon}\sigma^0\right) \\ \boldsymbol{\sigma}(\boldsymbol{\epsilon}^{-1}) &= \epsilon_p\left(i\boldsymbol{\sigma} + \frac{\epsilon_p\epsilon - i}{\epsilon^2}\boldsymbol{\epsilon}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} + \epsilon_p\hat{\epsilon}\sigma^0\right) \end{aligned} \tag{43}$$

The corresponding differential operators are

$$\begin{aligned} P^{\alpha\mu} &= (1, \nabla_s)i\partial_x^\mu = \partial_s^\alpha i\partial_x^\mu \\ P_s^{\alpha\mu} &= (\partial_\lambda^\mu, -\nabla_\lambda^\mu) = \partial_\lambda^{\alpha\mu} \end{aligned} \tag{44}$$

where $\partial_\lambda^\mu = (\frac{1}{\gamma^2}\frac{\partial}{\partial\beta}, -\nabla_s)$ and $-\nabla_\lambda^\mu = \boldsymbol{\beta}^\mu$. The transformed operators are given by

$$\begin{aligned} \partial_s^0(\boldsymbol{\epsilon}) &= \epsilon_p(1 + \boldsymbol{\epsilon}\cdot\nabla_s) \\ \nabla_s(\boldsymbol{\epsilon}) &= \epsilon_p\left(\nabla_s + \frac{\epsilon_p - 1}{\epsilon^2}\boldsymbol{\epsilon}\boldsymbol{\epsilon}\cdot\nabla_s + \epsilon_p\boldsymbol{\epsilon}\right) \\ \partial_\lambda^\mu(\boldsymbol{\epsilon}) &= \epsilon_p(\partial_\lambda^\mu - \boldsymbol{\epsilon}\cdot\boldsymbol{\beta}^\mu) \\ -\nabla_\lambda^\mu(\boldsymbol{\epsilon}) &= \boldsymbol{\beta}^\mu + \frac{\epsilon_p - 1}{\epsilon^2}\boldsymbol{\epsilon}\boldsymbol{\epsilon}\cdot\boldsymbol{\beta}^\mu - \epsilon_p\boldsymbol{\epsilon}\partial_\lambda^\mu \end{aligned} \tag{45}$$

The third and fourth relations come from the noncommutative extension of Maxwell equations which will follow. Using the differential operator, we can easily identified $x_s = \beta$ in the noncommutative extension of Yang-Mills field Eq. (14)

$$\begin{aligned} P'^\mu &= \epsilon_p\left(P^\mu + \boldsymbol{\epsilon}\cdot\mathbf{P}_s^\mu\right) - \tfrac{1}{2}q\epsilon_p(A^\mu(x, x_s) + \boldsymbol{\epsilon}\cdot\mathbf{A}_s^\mu(x, x_s)) \\ &= \epsilon_p(\epsilon + \boldsymbol{\tau}\cdot\nabla_s)P^\mu - \tfrac{1}{2}q\epsilon_p(\epsilon A^\mu(x, \beta) + \boldsymbol{\tau}\cdot\mathbf{A}_s^\mu(x, \beta)) \end{aligned} \tag{46}$$

where $P'^\mu = \frac{\partial}{\partial\vartheta}\hbar k'^\mu$ is used. The noncommutative extension of covariant derivative becomes

$$D_c^\mu = \epsilon_p(\epsilon + \boldsymbol{\tau}\cdot\nabla_s)\partial_x^\mu - \tfrac{1}{2}iq\epsilon_p(\epsilon A^\mu(x, \beta) + \boldsymbol{\tau}\cdot\mathbf{A}_s^\mu(x, \beta)) \tag{47}$$

We will continue the investigation of detail structure of the gauge fields in section 4 to calculate weak mixing angle. Main purpose of this section is to find an exact Lorentz invariant, confined noncommutative space-time structure.

Since the free motion does not provide Lorentz invariance, we investigate a dynamic motion by constructing noncommutative extension of commutative field theory. Up to now, the momentum and spin structures are studied separately. It will provide some messy notations for dynamic study. Since they can be easily distinguished by the Eq. (44) and Eq. (45), we use a view independent description by deleting all subscripts.

$$\begin{aligned} \partial^{\alpha\mu} &= (\partial^{\mu}, -\nabla^{\mu}) \\ A^{\alpha\mu} &= (A^{\mu}, \mathbf{A}^{\mu}) \\ J^{\alpha\mu} &= (J^{\mu}, \mathbf{J}^{\mu}) \end{aligned} \tag{48}$$

The fact that we have a noncommutative differential operator makes an easy extension of commutative field tensor such as

$$F_{\pm}^{\alpha\mu\beta\nu} = \tfrac{1}{2}(F^{\alpha\mu\beta\nu} \pm F^{\beta\mu\alpha\nu}) \tag{49}$$

where $\pm$ comes from the separation of field tensor in symmetric and in antisymmetric parts

$$\begin{aligned} F^{\alpha\mu\beta\nu} &= \partial^{\alpha\mu}A^{\beta\nu} - \partial^{\beta\nu}A^{\alpha\mu} \\ F^{\beta\mu\alpha\nu} &= \partial^{\beta\mu}A^{\alpha\nu} - \partial^{\alpha\nu}A^{\beta\mu} \end{aligned} \tag{50}$$

The symmetric part of field tensor is given by

$$F_{+}^{\alpha\mu\beta\nu} = \begin{pmatrix} F^{\mu\nu} & E_1^{\mu\nu} & E_2^{\mu\nu} & E_3^{\mu\nu} \\ E_1^{\mu\nu} & G_1^{\mu\nu} & B_3^{\mu\nu} & B_2^{\mu\nu} \\ E_2^{\mu\nu} & B_3^{\mu\nu} & G_2^{\mu\nu} & B_1^{\mu\nu} \\ E_3^{\mu\nu} & B_2^{\mu\nu} & B_1^{\mu\nu} & G_3^{\mu\nu} \end{pmatrix} \tag{51}$$

There are one isoscalar field tensor $F^{\mu\nu}$ and three isovector field tensors such as the electric $\mathbf{E}^{\mu\nu}$, the magnetic $\mathbf{B}^{\mu\nu}$, and the axial $\mathbf{G}^{\mu\nu}$. Each contains the electric field and the magnetic field components, respectively. The axial field tensor acts on spin polarization direction in particle view and acts on particle direction in spin view. The direction $\hat{\beta}$ of axial motion is actually same as that of spin polarization and that of particle motion as long as the combined system moves in 3D radial motion $\boldsymbol{\beta}$, i.e., the 1D radial isospin ϵ_{β}-motion. For 3D radial+angular motion $\boldsymbol{\beta}'$, i.e., 3D radial+angular isospin motion $\boldsymbol{\epsilon}$, the axial force does not follow same direction of external motion of particle. The axial force is a new feature of the confined noncommutative field theory.

$$\begin{aligned} F^{\mu\nu} &= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \\ \mathbf{E}^{\mu\nu} &= \tfrac{1}{2}(\partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu} + \nabla^{\nu}A^{\mu} - \nabla^{\mu}A^{\nu}) \\ \mathbf{B}^{\mu\nu} &= -\tfrac{1}{2}(\nabla^{\mu}\bar{\times}\mathbf{A}^{\nu} - \nabla^{\nu}\bar{\times}\mathbf{A}^{\mu}) \\ \mathbf{G}^{\mu\nu} &= -(\nabla^{\mu}\bar{\cdot}\mathbf{A}^{\nu} - \nabla^{\nu}\bar{\cdot}\mathbf{A}^{\mu}) \end{aligned} \tag{52}$$

The two new operators, $\bar{\times}$ and $\bar{\cdot}$, come from spin operations

$$\begin{aligned} (\mathbf{A}\bar{\times}\mathbf{B})_1 &\equiv \hat{\imath}(A_2B_3 + A_3B_2) \longrightarrow \boldsymbol{\sigma}\bar{\times}\boldsymbol{\sigma} = 0 \\ (\mathbf{A}\bar{\cdot}\mathbf{B})_1 &\equiv \hat{\imath}(A_1B_1) \longrightarrow \boldsymbol{\sigma}\bar{\cdot}\boldsymbol{\sigma}\bar{\cdot}\boldsymbol{\sigma} = \boldsymbol{\sigma} \end{aligned} \tag{53}$$

The detail form of the three vector field tensors follows same structure of $F^{\mu\nu}$. They are all antisymmetric. With the 4th rank field tensor, the noncommutative field equations become

$$
\begin{aligned}
\partial_{\alpha\mu} F_{\pm}^{\alpha\mu\beta\nu} &= J_{\pm}^{\beta\nu} \\
\partial_{\alpha\mu} \tilde{F}_{+}^{\alpha\mu\beta\nu} &= 0 \\
\partial_{\alpha\mu} F_{-}^{\beta\nu\gamma\rho} + \partial_{\beta\nu} F_{-}^{\gamma\rho\alpha\mu} + \partial_{\gamma\rho} F_{-}^{\alpha\mu\beta\nu} &= 0
\end{aligned}
\tag{54}
$$

where $J_{\pm}^{0\nu} = (\rho, \mathbf{J})_{\pm}$ and $\mathbf{J}_{\pm}^{\nu} = (-\vec{\rho}, -\vec{\mathbf{J}})_{\pm}$. $\frac{4\pi}{c}$ is set to 1. From the master equations, the dynamic motion in terms of four 2nd rank field tensors can be written as

$$
\begin{aligned}
\partial_{\mu} F^{\mu\nu} + \nabla_{\mu} \cdot \mathbf{E}^{\mu\nu} &= J^{\nu} \\
\partial_{\mu} \mathbf{E}^{\mu\nu} + \nabla_{\mu} \bar{\cdot} \mathbf{G}^{\mu\nu} + \nabla_{\mu} \bar{\times} \mathbf{B}^{\mu\nu} &= \mathbf{J}^{\nu} \\
\partial_{\mu} \tilde{F}^{\mu\nu} + \nabla_{\mu} \cdot \tilde{\mathbf{E}}^{\mu\nu} &= 0 \\
\partial_{\mu} \tilde{\mathbf{E}}^{\mu\nu} + \nabla_{\mu} \bar{\cdot} \tilde{\mathbf{G}}^{\mu\nu} + \nabla_{\mu} \bar{\times} \tilde{\mathbf{B}}^{\mu\nu} &= 0
\end{aligned}
\tag{55}
$$

To reproduce Maxwell equations can be done by the extraction

$$
\begin{aligned}
\vec{\partial} &= \boldsymbol{\sigma}\partial, \quad & \vec{\nabla} &= \boldsymbol{\sigma}\nabla \\
\vec{\phi} &= \boldsymbol{\sigma}\phi, \quad & \vec{\mathbf{A}} &= \boldsymbol{\sigma}\mathbf{A} \\
\vec{\rho} &= \boldsymbol{\sigma}\rho, \quad & \vec{\mathbf{J}} &= \boldsymbol{\sigma}\mathbf{J}
\end{aligned}
\tag{56}
$$

The noncommutative field equations provide a base to construct a corresponding noncommutative force equations which are necessary to find the missing term, the reason of breaking Lorentz invariance, by looking at the inertia structure of noncommutative dynamic motion. Since the charge q always follows with particle motion, we may define the charge with direction of motion

$$
\sigma^{\beta} q u^{\nu} = (\sigma^0 q u^{\nu}, \boldsymbol{\sigma} q u^{\nu}) \equiv (q^0 u^{0\nu}, \mathbf{q}\bar{\cdot}\mathbf{u}^{\nu}) = (q^0 u^{0\nu}, \hat{\beta} q_{\beta} u_{\beta}^{\nu} + \hat{\theta} q_{\theta} u_{\theta}^{\nu} + \hat{\varphi} q_{\varphi} u_{\varphi}^{\nu}) = q^{\beta\nu} \tag{57}
$$

A shorthand notation $q\mathbf{u}^{\nu} \equiv \mathbf{q}\bar{\cdot}\mathbf{u}^{\nu}$ is used for convenience. With the charge-with-motion definition, the 4th rank field tensor gives a noncommutative force equations

$$
f_{\pm}^{\alpha\mu} \equiv q_{\beta\nu} F_{\pm}^{\alpha\mu\beta\nu} \tag{58}
$$

The symmetric part of field and force is only treated in this paper. Following the force definition generates the detail structure of field force as in field equations, with the 2nd rank tensor field tensors,

$$
\begin{aligned}
f^{\mu} &= q^0 u_{\nu} F^{\mu\nu} - q\mathbf{u}_{\nu} \cdot \mathbf{E}^{\mu\nu} \\
\mathbf{f}^{\mu} &= q^0 u_{\nu} \mathbf{E}^{\mu\nu} - q\mathbf{u}_{\nu} \bar{\cdot} \mathbf{G}^{\mu\nu} - q\mathbf{u}_{\nu} \bar{\times} \mathbf{B}^{\mu\nu}
\end{aligned}
\tag{59}
$$

which clearly indicate that the three forces, the electric, the magnetic, and the axial forces, consist the non-Abeilan parts of noncommutative force. The non-Abelian velocity, $\mathbf{u}^{\nu}$, is given in Eq.(28) for momentum view and for spin view. From the noncommutative force, we find the exact momentum structure.

$$
\begin{aligned}
f_{+}^{\alpha\mu} &= \tfrac{1}{2} q_{\beta\nu} (\partial^{\alpha\mu} A^{\beta\nu} - \partial^{\beta\nu} A^{\alpha\mu} + \partial^{\beta\mu} A^{\alpha\nu} - \partial^{\alpha\nu} A^{\beta\mu}) \\
&= \tfrac{1}{2} (\partial^{\alpha\mu} q_{\beta\nu} A^{\beta\nu} - \partial^{\alpha\nu} q_{\beta\nu} A^{\beta\mu}) - \tfrac{1}{2} (q_{\beta\nu} \partial^{\beta\nu} A^{\alpha\mu} - q_{\beta\nu} \partial^{\beta\mu} A^{\alpha\nu}) \\
&= \tfrac{1}{2} (\partial^{\alpha\mu} q_{\beta\nu} A^{\beta\nu} - \partial^{\alpha\nu} q_{\beta\nu} A^{\beta\mu}) - \frac{d}{d\tau^{\nu}} \tfrac{1}{2} (q^{\nu} A^{\alpha\mu} - q^{\mu} A^{\alpha\nu}) \\
&\equiv \frac{d}{d\tau^{\nu}} (m^{\nu} u^{\alpha\mu} - m^{\mu} u^{\alpha\nu}) \\
&\equiv \frac{d}{d\tau^{\nu}} P_{-}^{\nu\alpha\mu}
\end{aligned}
\tag{60}
$$

where we use the analogy

$$qu_\nu\partial^\nu = \frac{d}{d\tau}q \longleftrightarrow q_{\beta\nu}\partial^{\beta\nu} = \frac{d}{d\tau^\nu}q^\nu \tag{61}$$

The conjugated momentum and potential follow

$$\begin{aligned}\frac{d}{d\tau^\nu}P^{\nu\alpha\mu}_{-c} &= \frac{d}{d\tau^\nu}\Big((m^\nu u^{\alpha\mu} + \tfrac{1}{2}q^\nu A^{\alpha\mu}) - (m^\mu u^{\alpha\nu} + \tfrac{1}{2}q^\mu A^{\alpha\nu})\Big)\\ &= \partial^{\alpha\mu}(\tfrac{1}{2}q_{\beta\nu}A^{\beta\nu}) - \partial^{\alpha\nu}(\tfrac{1}{2}q_{\beta\nu}A^{\beta\mu})\end{aligned} \tag{62}$$

Therefore, we define the noncommutative force equations which connect the field and the momentum

$$\frac{d}{d\tau^\nu}P^{\nu\alpha\mu}_{\mp} = q_{\beta\nu}F^{\alpha\mu\beta\nu}_{\pm} \tag{63}$$

with the 3rd rank tensor for noncommutative momentum

$$P^{\nu\alpha\mu}_{\mp} = (m^\nu u^{\alpha\mu} \mp m^\mu u^{\alpha\nu}) = (P^{\nu\mu}, \mathbf{P}^{\nu\mu})_\mp \tag{64}$$

It should be noted that the momentum structure comes from the requirement of noncommutative field equations, not arbitrary. Since the antisymmetric momentum behaves like a field, we may find a sourceless constraint equation. The whole noncommutative force equations can be summarized as

$$\begin{aligned}\frac{d}{d\tau^\nu}P^{\nu\alpha\mu}_{\mp} &= q_{\beta\nu}F^{\alpha\mu\beta\nu}_{\pm}\\ \frac{d}{d\tau^\nu}\tilde{P}^{\nu\alpha\mu}_{-} &= 0\\ \frac{d}{d\tau^\mu}P^{\nu\alpha\rho}_{+} + \frac{d}{d\tau^\nu}P^{\rho\alpha\mu}_{+} + \frac{d}{d\tau^\rho}P^{\mu\alpha\nu}_{+} &= 0\end{aligned} \tag{65}$$

The third rank tensor momentum is what we are looking for since it provides a way to invariance by deciding the structure of mass 4-vector

For the antisymmetric noncommutative momentum, $P^{\nu\alpha\mu}_{-} = m^\nu u^{\alpha\mu} - m^\mu u^{\alpha\nu}$,

$$m^\nu = \frac{1}{\sqrt{2}}\left(\frac{m^0}{\sqrt{1+\frac{2}{\gamma^2}}}, \hat{\beta}\frac{m_\beta}{\sqrt{1+\frac{2}{\gamma^2}}} + \hat{\theta}\frac{m_\theta}{\sqrt{2+\frac{1}{\gamma^2}}} + \hat{\varphi}\frac{m_\varphi}{\sqrt{2+\frac{1}{\gamma^2}}}\right) \tag{66}$$

and for the symmetric noncommutative momentum, $P^{\nu\alpha\mu}_{+} = m^\nu u^{\alpha\mu} + m^\mu u^{\alpha\nu}$,

$$m^\nu = \frac{1}{\sqrt{2}}\left(\frac{m^0}{\sqrt{3+\frac{2}{\gamma^2}}}, \hat{\beta}\frac{m_\beta}{\sqrt{3+\frac{2}{\gamma^2}}} + \hat{\theta}\frac{m_\theta}{\sqrt{2+\frac{3}{\gamma^2}}} + \hat{\varphi}\frac{m_\varphi}{\sqrt{2+\frac{3}{\gamma^2}}}\right) \tag{67}$$

By the normalized mass 4-vectors, we can restore the Lorentz invariance to the confined noncommutative momentum

$$P_{\mp\nu\alpha\mu}(\boldsymbol{\epsilon})P^{\nu\alpha\mu}_{\mp}(\boldsymbol{\epsilon}) = P_{\mp\nu\alpha\mu}P^{\nu\alpha\mu}_{\mp} = 2(m_\mp\cdot m_\mp u_\alpha\cdot u^\alpha \mp m_\mp\cdot u_\alpha m_\mp\cdot u^\alpha) = m_0^2c^2 \tag{68}$$

with

$$m_0^2 = m^{o2} - (m_\beta^2 + m_\theta^2 + m_\varphi^2) = m^{o2} - \mathbf{m}^2 \tag{69}$$

The mass normalization plays a key role toward invariance.

A corresponding noncommutative space-time structure can be found from the covariant momentum. Since the noncommutative momentum now holds for the whole range of external velocity $\beta = [0, 1]$ and isospin velocity $\epsilon = [-1, 1]$, we use the de Broglie 4-vectors in Eq. (29). The de Broglie 4-vectors and the experimental fact of the wave-particle duality in keV scale, for example, the classical compton scattering, suggest the wave effect of particle motion may be included in the real space-time structure if it is connected to spin as in Eq. (29). We first assume that the wave effect of particle is added on the Minkowski space-time ,$x^\mu = (ct, \mathbf{r})$, in parallel with the noncommutative motion P'^μ in Eq. (4)

$$x_c^\mu = x^\mu + \frac{\lambda}{mc} p_\lambda^\mu \tag{70}$$

where $p_\lambda^\mu = (E_\lambda/c, \mathbf{p}_\lambda)$ represents a wave variable which is assumed to independent on particle variable x^μ. The relation $\lambda mc = h$ connects the two constants λ and mc. We may consider p_λ^μ as a momentum of noncommutative field in [5]. The supposed space-time 4-vector x_c^μ gives a corresponding proper time

$$d\tau_{c\pm} = d\tau \pm \frac{\lambda}{mc^3} d\varepsilon_\lambda \tag{71}$$

The proper energy $d\varepsilon_\lambda = dE_\lambda/\gamma\beta$ is connected to an orthogonal condition $\beta_\mu \beta_\lambda^{-1^\mu} = 1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}_\lambda^{-1} = 0$ and to the inverse of group velocity $\boldsymbol{\beta}_\lambda^{-1} = cd\mathbf{p}_\lambda/dE_\lambda$. The proper time of particle becomes as usual $d\tau = dt/\gamma$. Lorentz transformation for particle also naturally follows same as in special relativity. But, in contrast, the pure boost for p_λ^μ exhibits a spin nature, hopefully,

$$p'_{\lambda\mu} p_\lambda'^{\,\mu} = -p_{\lambda_\mu} p_\lambda^\mu \tag{72}$$

with

$$p'_{\lambda\mu} = ie^{\theta\sigma} p_\lambda^\mu = (i + \sigma^2(\gamma\beta - i) + \sigma\gamma) p_\lambda^\mu \tag{73}$$

where $\tanh\theta = 1/\beta$, $\cosh\theta = -i\gamma\beta$, and $\sinh\theta = -i\gamma$ come from an infinitesimal limit $\delta(\beta_\lambda^{-1}) \to 0$. σ is given in standard texts. Since there are two independent variables, it is natural to take a directional derivative to get a combined 4-momentum with direction cosine $\gamma(\epsilon) = \cosh\vartheta$ and $\gamma(\epsilon)\epsilon = \sinh\vartheta$

$$P_c^\mu(\epsilon) = \gamma(\epsilon)(P^\mu + \epsilon P_\lambda^\mu) \tag{74}$$

The directional constant $\epsilon = \tanh\vartheta = [-1, 1]$ now absorbs $\pm$ signs in total proper time. The particle momentum and the wave related momentum become

$$\begin{aligned} P^\mu &= mc\gamma(1, \boldsymbol{\beta}) \\ P_\lambda^\mu &= mc\gamma(\beta, \hat{\beta}) \end{aligned} \tag{75}$$

Since the wave related momentum P_λ^μ is exactly equal to the longitudinal spin polarization vector

$$s^\mu = \frac{1}{mc} P_\lambda^\mu = \gamma(\beta, \hat{\beta}) \tag{76}$$

we can safely use the supposed space-time x_c^μ as the noncommutative space-time. For 3D isospin motion $\boldsymbol{\epsilon} = \epsilon\boldsymbol{\tau}$, we define

$$x^\mu(\boldsymbol{\epsilon}) = \epsilon_p \left(x^\mu + \frac{\lambda}{mc} \epsilon \boldsymbol{\tau} \cdot \mathbf{p}_\lambda^\mu \right) \tag{77}$$

Since the non-Ableian wave vector $\mathbf{p}_\lambda^\mu = \hat{\beta} p_{\lambda_\beta}^{\ \mu} + \hat{\theta} p_{\lambda_\theta}^{\ \mu} + \hat{\varphi} p_{\lambda_\varphi}^{\ \mu}$ generates non-Abelian spin polarization Eq. (28) by differentiation, we find

$$\begin{aligned} p_{\lambda_\beta}^{\ \mu} &= (E_\lambda/c, \mathbf{p}_{\lambda_\beta}) = (E_\lambda/c, \hat{\beta} p_{\lambda_\beta}) \\ p_{\lambda_\theta}^{\ \mu} &= (0, \mathbf{p}_{\lambda_\theta}) = (0, \hat{\theta} p_{\lambda_\theta}) \\ p_{\lambda_\varphi}^{\ \mu} &= (0, \mathbf{p}_{\lambda_\varphi}) = (0, \hat{\varphi} p_{\lambda_\varphi}) \end{aligned} \tag{78}$$

Therefore, the noncommutative space-time structure in Eq. (1) becomes

$$[x^\mu(\boldsymbol{\epsilon}), x^\nu(\boldsymbol{\epsilon})] = i\theta^{\mu\nu} = i\epsilon_p^2 \epsilon^2 \left(\frac{\lambda}{mc} \right)^2 2\boldsymbol{\tau} \cdot \mathbf{p}_\lambda^\mu \times \mathbf{p}_\lambda^\nu \tag{79}$$

The explicit form of $\theta^{\mu\nu}$ is given by

$$\theta^{\mu\nu} = \epsilon_p^2 \epsilon^2 \left(\frac{\lambda}{mc} \right)^2 \frac{2}{c} \begin{pmatrix} 0 & 0 & \tau_\varphi E_\lambda p_{\lambda_\theta} & -\tau_\beta E_\lambda p_{\lambda_\varphi} \\ 0 & 0 & \tau_\varphi p_{\lambda_\beta} p_{\lambda_\theta} & -\tau_\theta p_{\lambda_\beta} p_{\lambda_\varphi} \\ -\tau_\varphi E_\lambda p_{\lambda_\theta} & -\tau_\varphi p_{\lambda_\beta} p_{\lambda_\theta} & 0 & \tau_\beta p_{\lambda_\theta} p_{\lambda_\varphi} \\ \tau_\beta E_\lambda p_{\lambda_\varphi} & \tau_\theta p_{\lambda_\beta} p_{\lambda_\varphi} & -\tau_\beta p_{\lambda_\theta} p_{\lambda_\varphi} & 0 \end{pmatrix} \tag{80}$$

with $\epsilon = [-1, 1]$, $\epsilon_p = 1/\sqrt{1-\epsilon^2}$, and

$$\tau_\beta = \frac{1}{\sqrt{3}}(\tau_1 + \tau_2 + \tau_3) \ , \quad \tau_\theta = \frac{1}{\sqrt{6}}(\tau_1 + \tau_2 - 2\tau_3) \ , \quad \tau_\varphi = \frac{1}{\sqrt{2}}(\tau_2 - \tau_1) \tag{81}$$

The Lorentz invariance through mass normalization, Eq. (66)$\sim$ Eq. (69), requires a description of point mass instead of pure space-time consideration Eq. (1). For this purpose, we change notation

$$\mathbf{x}_s^\mu = \frac{\lambda}{mc} \mathbf{p}_\lambda^\mu \tag{82}$$

The the noncommutative space-time Eq. (77) can be rewritten as

$$\begin{aligned} x^\mu(\boldsymbol{\epsilon}) &= \epsilon_p \left(x^\mu + \epsilon \boldsymbol{\tau} \cdot \mathbf{x}_s^\mu \right) \\ x_s^\mu(\boldsymbol{\epsilon}) &= \epsilon_p \left(x_s^\mu + \epsilon \boldsymbol{\tau} \cdot \mathbf{x}^\mu \right) \end{aligned} \tag{83}$$

where the second equation corresponds to spin polarization term in Eq. (9). The non-Abelian vector parts of noncommutative space-time follow

$$\begin{aligned} \mathbf{x}^\mu(\boldsymbol{\epsilon}) &= \epsilon_p \left(\mathbf{x}^\mu + \frac{\epsilon_p - 1}{\epsilon^2} \boldsymbol{\epsilon}\boldsymbol{\epsilon} \cdot \mathbf{x}^\mu + \epsilon_p \boldsymbol{\epsilon} x_s^\mu \right) \\ \mathbf{x}_s^\mu(\boldsymbol{\epsilon}) &= \epsilon_p \left(i\mathbf{x}_s^\mu + \frac{\epsilon_p \epsilon - i}{\epsilon^2} \boldsymbol{\epsilon}\boldsymbol{\epsilon} \cdot \mathbf{x}_s^\mu + \epsilon_p \hat{\epsilon} x^\mu \right) \end{aligned} \tag{84}$$

In parallel with the third rank noncommutative momentum, $P^{\nu\alpha\mu} = m^\nu u^{\alpha\mu} - m^\mu u^{\alpha\nu}$, we define the noncommutative extension of covariant mass point or point mass

$$(mx)^{\nu\alpha\mu} \equiv m^\nu x^{\alpha\mu} - m^\mu x^{\alpha\nu} \tag{85}$$

with the non-Abelian transformation

$$(mx)^{\nu\alpha\mu}(\boldsymbol{\epsilon}) = m^\nu x^{\alpha\mu}(\boldsymbol{\epsilon}) - m^\mu x^{\alpha\nu}(\boldsymbol{\epsilon}) \tag{86}$$

where $\boldsymbol{\epsilon} = \epsilon\boldsymbol{\tau}$. The most general form of the confined noncommutative space-time-mass structure is

$$[(mx)^{\nu\alpha\mu}(\boldsymbol{\epsilon}), (mx)^{\lambda\beta\rho}(\boldsymbol{\epsilon})] \tag{87}$$

The isoscalar component, $\alpha = \beta = 0$ reads

$$\begin{aligned}[(mx)^{\nu\mu}(\boldsymbol{\epsilon}), (mx)^{\lambda\rho}(\boldsymbol{\epsilon})] &= [m^\nu x^\mu(\boldsymbol{\epsilon}) - m^\mu x^\nu(\boldsymbol{\epsilon}), m^\lambda x^\rho(\boldsymbol{\epsilon}) - m^\rho x^\lambda(\boldsymbol{\epsilon})] \\ &= i\epsilon_p^2\epsilon^2 2\boldsymbol{\tau}\cdot(m^\nu \mathbf{x}_s^\mu - m^\mu \mathbf{x}_s^\nu)\times(m^\lambda \mathbf{x}_s^\rho - m^\rho \mathbf{x}_s^\lambda)\end{aligned} \tag{88}$$

Projection to space-time coordinate becomes

$$[x_R^\mu(\boldsymbol{\epsilon}), x_R^\nu(\boldsymbol{\epsilon})] = i\theta_R^{\mu\nu} = i\epsilon_p^2\epsilon^2 2\boldsymbol{\tau}\cdot\mathbf{x}_{s_R}^\mu \times \mathbf{x}_{s_R}^\nu \tag{89}$$

with the normalized space-time

$$\begin{aligned} x_R^\mu(\boldsymbol{\epsilon}) &= x^\mu(\boldsymbol{\epsilon}) - m^\mu \frac{m\cdot x(\boldsymbol{\epsilon})}{m\cdot m} \\ \mathbf{x}_{s_R}^\mu &= \mathbf{x}_s^\mu - m^\mu \frac{m\cdot \mathbf{x}_s}{m\cdot m}\end{aligned} \tag{90}$$

The above equation describes the confined noncommutative structure in a space-time mass point. Note that to maintain Lorentz invariance it is neceesary to weight the normalized space-time by

$$x_R'^{\alpha\mu}(\boldsymbol{\epsilon}) = \frac{\sqrt{2m\cdot m}}{m_0} x_R^{\alpha\mu}(\boldsymbol{\epsilon}) \tag{91}$$

The mass 4-vectors are given in Eq. (66) and Eq. (67) with a change of normalization value in symmetric case

$$\begin{aligned} m_-\cdot m_- &= \frac{1}{2}\left(\frac{m^{o2} - m_\beta^2}{1+\frac{2}{\gamma^2}} - \frac{m_\theta^2 + m_\varphi^2}{2+\frac{1}{\gamma^2}}\right) \xrightarrow{\beta\to 0} \frac{1}{6}m_0^2 \\ m_+\cdot m_+ &= \frac{1}{2}\left(\frac{m^{o2} - m_\beta^2}{5+\frac{2}{\gamma^2}} - \frac{m_\theta^2 + m_\varphi^2}{2+\frac{5}{\gamma^2}}\right) \xrightarrow{\beta\to 0} \frac{1}{20}m_0^2 \end{aligned} \tag{92}$$

The normalized noncommutative parameter for isoscalar part is given by

$$\theta_R^{\mu\nu} = \epsilon_p^2\epsilon^2\lambda^2 2\boldsymbol{\tau}\cdot\hat{\mathbf{x}}_{s_R}^\mu \times \hat{\mathbf{x}}_{s_R}^\nu \tag{93}$$

where the dimensionless unit is used to evaluate the magnitude of normalized noncommutative parameter

$$\theta \equiv (\epsilon_p \epsilon \lambda \sqrt{2})^2 \tag{94}$$

We note that the scale of noncommutativity is not fixed but varied with responding dynamics. For example, at the electroweak scale in section 4, we find the isospin velocity and the invariant mass for isoscalar motion

$$\epsilon = 1/2 \ , \qquad \lambda = \frac{h}{mc} \cong (168\ GeV)^{-1} \tag{95}$$

which gives

$$\theta \cong (206\ GeV)^{-2} \tag{96}$$

At the keV scale, the classical compton scattering region as in section 3, the isospin velocity ϵ might be expected to be near zero since the spin polarization effect in the scale is typically quite small one. Therefore we can expect that θ_R is not a fixed constant for whole scale. The noncommutative momentum structure also can be found both in general form

$$[P^{\nu\alpha\mu}(\boldsymbol{\epsilon}), P^{\lambda\beta\rho}(\boldsymbol{\epsilon})] = [m^\nu u^{\alpha\mu}(\boldsymbol{\epsilon}) - m^\mu u^{\alpha\nu}(\boldsymbol{\epsilon}), m^\lambda u^{\beta\rho}(\boldsymbol{\epsilon}) - m^\rho u^{\beta\lambda}(\boldsymbol{\epsilon})] \tag{97}$$

and in projection to velocity space , for $\alpha = \beta = 0$,

$$[P_R^\mu(\boldsymbol{\epsilon}), P_R^\nu(\boldsymbol{\epsilon})] = i\eta_R^{\mu\nu} = i\epsilon_p^2 \epsilon^2 2\boldsymbol{\tau} \cdot \mathbf{P}_{s_R}^\mu \times \mathbf{P}_{s_R}^\nu \tag{98}$$

with the normalized momentum

$$\begin{aligned} P_R^\mu(\boldsymbol{\epsilon}) &= P^\mu(\boldsymbol{\epsilon}) - m^\mu \frac{m \cdot P(\boldsymbol{\epsilon})}{m \cdot m} \\ \mathbf{P}_{s_R}^\mu &= \mathbf{P}_s^\mu - m^\mu \frac{m \cdot \mathbf{P}_s}{m \cdot m} \\ P^\mu(\boldsymbol{\epsilon}) &\equiv \sqrt{2m \cdot m}\ u^\mu(\boldsymbol{\epsilon}) \\ \mathbf{P}_s^\mu &\equiv \sqrt{2m \cdot m}\ \mathbf{u}_s^\mu \end{aligned} \tag{99}$$

The normalized momentum maintains Lorentz invariance

$$P_{R\alpha\mu}(\boldsymbol{\epsilon}) P_R^{\alpha\mu}(\boldsymbol{\epsilon}) = P_{R\alpha\mu} P_R^{\alpha\mu} = m_0^2 c^2 \tag{100}$$

The notable thing to the noncommutative extension of momentum conservation is the conservation of momentum-spin polarization.

$$P^{\nu\alpha\mu}(\boldsymbol{\epsilon}_1) + Q^{\nu\alpha\mu}(\boldsymbol{\epsilon}_2) = P'^{\nu\alpha\mu}(\boldsymbol{\epsilon}'_1) + Q'^{\nu\alpha\mu}(\boldsymbol{\epsilon}'_2) \tag{101}$$

where P, Q for initial momentum and P', Q' for final momentum. Since total 256 components contribute the conservation law, it might be reasonable to take the projected momentum for conservation law which has total 64 components.

$$P_R^{\alpha\mu}(\boldsymbol{\epsilon}_1) + Q_R^{\alpha\mu}(\boldsymbol{\epsilon}_2) = P_R'^{\alpha\mu}(\boldsymbol{\epsilon}'_1) + Q_R'^{\alpha\mu}(\boldsymbol{\epsilon}'_2) \tag{102}$$

From the noncommutative momentum-spin conservation law, we can completely trace both the momentum change in the external $\boldsymbol{\beta}$-transformation and the spin change in the internal isospin velocity $\boldsymbol{\epsilon} = \epsilon\boldsymbol{\tau}$-transformation.

In brief, the inclusion of spin polarization in momentum by following the Seiberg-Witten map [3,12] requires changing the concept of space-time point from pure geometric one to space-time mass point to maintain the Lorentz invariance under boost transformation both in external and in internal. The important consequence that the theory holds now for whole range of both $\beta = [0, 1]$ and $\epsilon = [-1, 1]$ needs to be tested by experiment, especially in low energy scale since the non-Abelian nature should exists in near rest state $\beta = 0$. Most simple test may be possible in classical compton scattering experiment with using a simple formula, the noncommutative extension of energy-momentum conservation law.

III. NONCOMMUTATIVE COMPTON SCATTERING

The classical compton scattering in a coplane of the incident photon k^μ, the scattered photon k'^μ, the initial momentum P^μ and the final momentum P'^μ of a particle follows the energy-momentum conservation

$$k^\mu + P^\mu = k'^\mu + P'^\mu \tag{103}$$

For a fermion and the unpolarized photon, we extend the law by following the noncommutative relation Eq. (9)

$$k^\mu + P^\mu(\boldsymbol{\epsilon}) = k'^\mu + P'^\mu(\boldsymbol{\epsilon}) \tag{104}$$

with

$$P^\mu(\boldsymbol{\epsilon}) = (E^\mu(\boldsymbol{\epsilon})/c, \mathbf{P}^\mu(\boldsymbol{\epsilon})) = \epsilon_p(P^\mu + \epsilon\boldsymbol{\tau}\cdot mc\mathbf{s}^\mu) \tag{105}$$

and $\epsilon_p = 1/\sqrt{1-\epsilon^2}$. The non-Abelian spin polarization $\mathbf{s}^\mu$, given in Eq. (28), can be rewritten with spin notations

$$\mathbf{s}^\mu = \hat{s}_\beta(\mathbf{s}_\beta\cdot\boldsymbol{\beta}, \mathbf{s}_\beta) + \hat{s}_\theta(0, \mathbf{s}_\theta) + \hat{s}_\varphi(0, \mathbf{s}_\varphi) \tag{106}$$

Lorentz invariance to the isoscalar scattering process requires to take only the β-component 4-vector.

$$P_\mu(\epsilon)P^\mu(\epsilon) = \epsilon_p^2(P_\mu + \epsilon\; mc\; s_{\beta_\mu})(P^\mu + \epsilon\; mc\; s_\beta^\mu) = m^2c^2 \tag{107}$$

with $s^\mu = s_\beta^\mu = \gamma(\beta, \hat{s})$, $\hat{s} = \hat{\beta}$. Then the transformed momentum is simplified

$$P^\mu(\epsilon) = (E(\epsilon)/c, \mathbf{P}(\epsilon)) = \epsilon_p(P^\mu + \epsilon\; mc\; s^\mu) \tag{108}$$

Let's adjust the initial spin direction to the direction of incident photon with an angle ϕ such as $\hat{s}\cdot\hat{k} = \cos\phi$. Then, the initially rest and the finally scattered momentum become

$$\begin{aligned} P^\mu(\epsilon) &= (E(\epsilon)/c, \mathbf{P}(\epsilon)) = mc\epsilon_p(1, \epsilon\hat{s}) \\ P'^\mu(\epsilon) &= (E'(\epsilon)/c, \mathbf{P}'(\epsilon)) = mc\epsilon_p\gamma(1+\epsilon\beta', \boldsymbol{\beta}' - \epsilon\hat{s}') \end{aligned} \tag{109}$$

where we assume the isospin velocity ϵ does not change in this low energy scale. The modified energy-momentum conservation is

$$\begin{aligned} k^0 + E(\epsilon) &= k'^0 + E'(\epsilon) \\ \mathbf{k} + \mathbf{P}(\epsilon) &= \mathbf{k}' + \mathbf{P}'(\epsilon) \end{aligned} \tag{110}$$

Using the Lorentz invariance, we find

$$\lambda' - \lambda = \frac{h}{\epsilon_p mc}(1-\cos\vartheta) + \epsilon\hat{s}\cdot(\lambda'\hat{k} - \lambda\hat{k}') \tag{111}$$

which is independent on the final spin. The initial spin in opposite direction changes the sign only

$$\lambda' - \lambda = \frac{h}{\epsilon_p mc}(1-\cos\vartheta) - \epsilon\hat{s}\cdot(\lambda'\hat{k} - \lambda\hat{k}') \tag{112}$$

Then we can extract the spin dependent terms by subtracting the two equations

$$\tan\phi = \frac{1}{\sin\vartheta}\left(\cos\vartheta - \frac{\lambda'}{\lambda}\right) \tag{113}$$

where $\hat{s}\cdot\hat{k}' = \cos(\phi+\vartheta)$ and $\hat{k}\cdot\hat{k}' = \cos\vartheta$ are used. Therefore, the equation predicts that, for given energy of the initial photon, the scattering angle ϑ changes by tuning the spin polarization angle ϕ of an initially rest fermion. To test the noncommutative structure how to hold in another scale, the electroweak mixing angle is calculated in the next section.

IV. WEAK MIXING ANGLE

During the investigation of the noncommutative space-time mass point structure, we observed that the weak mixing angle could be calculated by comparing with the covariant derivative even though the derived gauge field structure is not fully understood. Here a simple process to calculate the weak parameters without fixing scale follows. The covariant derivative in Eq. (47) is given by

$$D_c^\mu(\boldsymbol{\epsilon}) = \epsilon_p(\epsilon + \boldsymbol{\tau}\cdot\nabla_s)\partial_x^\mu - \tfrac{1}{2}iq\epsilon_p(\epsilon A^\mu + \boldsymbol{\tau}\cdot\mathbf{A}_s^\mu) \tag{114}$$

We consider only the gauge fields to the purpose. The couplings may be identified as

$$\begin{aligned} g' &= \epsilon_p\epsilon q \\ g &= \epsilon_p q \\ \tan\theta_W &= \epsilon \end{aligned} \tag{115}$$

Therefore, the weak mixing angle can be written as

$$\sin^2\theta_W = \frac{\epsilon^2}{1+\epsilon^2} \tag{116}$$

The isospin velocity ϵ can be decided from the wave equation Eq. (54).

$$0 \equiv J_+^{\beta\nu} = \partial_{\alpha\mu}F_+^{\alpha\mu\beta\nu} = \partial_{\alpha\mu}\tfrac{1}{2}(\partial^{\alpha\mu}A^{\beta\nu} - \partial^{\beta\nu}A^{\alpha\mu} + \partial^{\beta\mu}A^{\alpha\nu} - \partial^{\alpha\nu}A^{\beta\mu}) \tag{117}$$

Since the second, third, and the last terms are related to gauge fixing condition, we drop the three terms and extract the first term as a wave equation.

$$\partial_{\alpha\mu}\partial^{\alpha\mu}A^{\beta\nu} = (\partial_\mu\partial^\mu - \nabla_\mu\cdot\nabla^\mu)A^{\beta\nu} = 0 \tag{118}$$

It is necessary to make a noncommutative extension of the wave equation in spin view as in the third and fourth relations of Eq. (45) to find ϵ.

$$\partial_{\alpha\mu}(\boldsymbol{\epsilon}')\partial^{\alpha\mu}(\boldsymbol{\epsilon})A^{\beta\nu} = [\partial_\mu(\boldsymbol{\epsilon}')\partial^\mu(\boldsymbol{\epsilon}) - \nabla_\mu(\boldsymbol{\epsilon}')\cdot\nabla^\mu(\boldsymbol{\epsilon})]A^{\beta\nu}(\beta) = 0 \tag{119}$$

where $\partial^\mu = (\frac{1}{\gamma^2}\frac{\partial}{\partial\beta}, -\nabla_s)$. The difference of ϵ comes from the observation

$$\boldsymbol{\sigma}\times\boldsymbol{\sigma}\phi = 2i\boldsymbol{\sigma}\phi \longleftrightarrow \nabla_s\times\nabla_s\phi = -2\boldsymbol{\beta}\times\nabla_s\phi \tag{120}$$

for arbitrary function $\phi = \phi(\boldsymbol{\beta})$. It follows a divergence equation.

$$\nabla_s(\epsilon+2)\cdot\mathbf{B}_s = (\nabla_s + (\epsilon+2)\boldsymbol{\beta})\cdot(\nabla_s + \epsilon\boldsymbol{\beta})\times\mathbf{A}_s = 0 \tag{121}$$

From the dependence of time component and the spatial component

$$\left(\partial_\tau + 2\beta + \frac{1}{\gamma^2\beta}\right)\nabla_s - \left(\nabla_s + 2\boldsymbol{\beta} + \hat{\beta}\frac{1}{\gamma^2\beta}\right)\partial_\tau = 0 \tag{122}$$

we can write the isoscalar wave equation

$$\partial_\mu(\epsilon+2)\partial^\mu(\epsilon)A^{\beta\nu}(\beta) = [\partial_\mu\partial^\mu - 2m(\epsilon+1)\beta_\mu\partial^\mu + m^2\epsilon(\epsilon-1)\beta_\mu\beta^\mu]A^{\beta\nu}(\beta) = 0 \tag{123}$$

with recovering the space-time units in differential operator. The longitudinally polarized spin is assumed for simplicity. In this isoscalar case, ϵ is determined from the mass function.

$$m^2(\epsilon,\beta) = m^2\epsilon(\epsilon-1)(1-\beta^2) \tag{124}$$

The local minimum of the mass function at $\beta = 0$ can be found in

$$\frac{d}{d\epsilon}m^2(\epsilon) = m^2(2\epsilon-1) = 0 \tag{125}$$

The minimum mass of the field is

$$m^2_{min} = -\frac{1}{4}m^2 \qquad at \;\; \epsilon = \frac{1}{2} \tag{126}$$

Therefore, the weak mixing angle for the isoscalar wave equation becomes

$$\sin^2\theta_W = \frac{\epsilon^2}{1+\epsilon^2} = \frac{1}{5} = 0.200 \tag{127}$$

The magnitude has a big difference, as expected, with $\sim$0.2315 from data and the predictions of MSSM [15]. A correction may be found in the isovector wave equation Eq. (119). Since the process is independent on the isospin rotation of vector fields, we may take an average to determine the order of invariant mass

$$m = 2|m_{min}| \cong 2\,\frac{1}{3}(2m_W + m_z) \cong 168\ GeV \tag{128}$$

We summarize the unusual results from the simple calculation process: 1. The weak mixing angle is decided without cut-off scale. 2. The local minimum of mass comes from the wave equation of gauge fields in spin view.

V. CONCLUSIONS

By following the Seiberg-Witten map, Lorentz invariance of a noncommutative fermion is maintained under both external and internal boost transformations. We find the free parameter ϵ in theory can be determined by the corresponding dynamics. In electroweak scale, the parameter is decided at the local minimum of field mass function and used to calculate the weak mixing angle. For low energy scale, we expect the parameter might be determined by an experimental test of a proposed classical compton scattering formula which includes a noncommutative effect.

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